

Geometric Annuities on the HP-12C

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The present value, at interest rate $i\%$ per period, of a geometrically increasing annuity with payments $1, (1+g), (1+g)^2, \dots, (1+g)^{n-1}$ paid at the end of each period, is $(1-(1+j)^{-n})/j/(1+g)$ where $j=(i-g)/(1+g)$. This is often written as $(1-(1+j)^{-n})/(i-g)$. Either way, the formula fails when $j=0$ (i.e. $i=g$), whereas it is equivalent to $USPV(j\%,n)/(1+g)$, which shows that for $j=0$ the value is just $n/(1+g)$. $j\%$ can be calculated in just 6 lines on the HP-12C, with $i\%$ and $g\%$ in Y and X respectively: $\boxed{-} \boxed{1} \boxed{g} \boxed{LSTx} \boxed{\%} \boxed{+} \boxed{\div}$. Then we can just use PV (with FV=0) to compute value of the increasing annuity, with END mode and $PMT=1/(1+g)$ which is what the payment rate *would* have been at time zero. For payment mode BEG we use the same $j\%$ but just need $PMT=1$. For computing the FV (with PV=0) we can also use the same $j\%$ and for mode END we have $PMT=(1+g)^{n-1}$ and for BEG mode $PMT=(1+g)^n$, and these are just payment rates at time n (notional for BEG mode), so it's pretty easy to remember. There are many applications of this - for example graduated payment mortgages, computing the present value of capital payments included in level annuity repayments, defined benefit pensions and property appraisal.

1. Property Appraisal

I give an example taken from Tirone's book on Real Estate [1] pages 263-269. The 45 line program given on page 267 is reduced to 35 lines and now works for $i=g$ and also preserves the original X value in the stack. In fact I explicitly added 5 lines (16,17,30,32 and 33) to add the feature of preserving X and also make sure that the value in PV at the end is the value without reversion, at the same time preserving PMT.

Keystrokes	Display	Keystrokes	Display	Keystrokes	Display
$\boxed{f} \boxed{P/R}$		\boxed{FV}	12 - 15	$\boxed{+}$	25 - 40
$\boxed{f} \boxed{CLEAR} \boxed{PRGM}$	00 -	\boxed{PV}	13 - 13	$\boxed{\div}$	26 - 10
$\boxed{g} \boxed{END}$	01 - 43 8	$\boxed{g} \boxed{LSTx}$	14 - 43 36	$\boxed{RCL} \boxed{n}$	27 - 45 11
$\boxed{RCL} \boxed{2}$	02 - 45 2	$\boxed{\div}$	15 - 10	$\boxed{y^x}$	28 - 21
$\boxed{RCL} \boxed{0}$	03 - 45 0	\boxed{CHS}	16 - 16	$\boxed{1}$	29 - 1
$\boxed{-}$	04 - 30	\boxed{PV}	17 - 13	$\boxed{x \geq y}$	30 - 34
$\boxed{1}$	05 - 1	$\boxed{1}$	18 - 1	$\boxed{-}$	31 - 30
$\boxed{g} \boxed{LSTx}$	06 - 43 36	$\boxed{RCL} \boxed{1}$	19 - 45 1	$\boxed{RCL} \boxed{PV}$	32 - 45 13
$\boxed{\%}$	07 - 25	$\boxed{\%}$	20 - 25	$\boxed{x \geq y}$	33 - 34
$\boxed{+}$	08 - 40	$\boxed{+}$	21 - 40	$\boxed{\div}$	34 - 10
$\boxed{\div}$	09 - 10	$\boxed{1}$	22 - 1	$\boxed{g} \boxed{GTO} \boxed{00}$	35 - 43,33 00
\boxed{i}	10 - 12	$\boxed{RCL} \boxed{2}$	23 - 45 2	$\boxed{f} \boxed{P/R}$	
\boxed{CLx}	11 - 35	$\boxed{\%}$	24 - 25		

I need to ensure END mode, so added that to the program. Tirone puts that in instruction 1 on page 267, but doesn't need it as the program there doesn't use TVM. The formula (easily the most complex in the book), which is a way of capitalizing income and also allows for re-sale ("reversion") of the property, is:

Present value = $NOI * USPV(j\%,n)/(1+g)/(1-(1+k)^{-n})$ where $j\%$ is as above, $k=(i-gv)/(1+gv)$, g is the growth rate for the NOI (Net Operating Income), and gv the growth rate of the PV itself. I evaluate the k -factor in the denominator explicitly, as in Tirone's program, so that the user can have $gv\%=-100$ and run the problem without reversion. This means the program can even be used to evaluate the simpler formula given on page 262. However note that the value without reversion is now *always* available at the end in PV.

The three rates are stored on Registers R_0 ($g\%$), R_1 ($gv\%$) and R_2 ($i\%$). n and NOI are stored in \boxed{n} and \boxed{PMT} , and are preserved.

Note, $LASTX = 1 - (1+k)^{-n}$, the denominator in the formula. PV is the numerator, or the value without reversion. i holds the net interest rate used in valuing the NOI.

Examples from pages 268/269 of Tirone.

First with $g=gv$ demonstrate how the value does not depend upon n . In this case $j=k$ in the formula and it reduces to: Present value = $NOI/(i-g)$. 3 $\boxed{STO}0$ (g) 3 $\boxed{STO}1$ (gv) 15 $\boxed{STO}2$ (i) 10 \boxed{n} 45000 \boxed{PMT} $\boxed{R/S}$ → 375,000.00 \boxed{RCL} \boxed{PV} → 250,4276.66

Try 2 different holding periods: 1 \boxed{n} $\boxed{R/S}$ → 375,000.00 \boxed{RCL} \boxed{PV} → 335,869

100 \boxed{n} $\boxed{R/S}$ → 375,000.00 \boxed{RCL} \boxed{PV} → 6.14. The longer the term the less impact of the reversion. The same again, but at a lower valuation interest rate: 10 \boxed{n} 13 \boxed{i} $\boxed{R/S}$ → 450,000.00, 100 \boxed{n} $\boxed{R/S}$ → 450,000.00

This values a \$60,000 p.a. lease, increasing at 3% p.a., over a 5 year holding period, at two different valuation rates:

3 $\boxed{STO}0$ (g), 100 \boxed{CHS} $\boxed{STO}1$ (gv), 10 $\boxed{STO}2$ (i), 5 \boxed{n} 60000 \boxed{PMT} $\boxed{R/S}$ → 240,156.01
20 $\boxed{STO}2$ $\boxed{R/S}$ → 88,510.74. \boxed{RCL} \boxed{i} at the end to see the "net" interest rate used in valuing the NOI.

[1]"Professional Real Estate Problem Solving Using the HP 12C", Second Edition, 1997, published and written by John A. Tirone. 302pp. A4. Very well worked examples.

2. Valuing Interest payments in an annuity

This computes the PV of the interest payments in an annuity - as the PV of all payments less the PV of the capital payments. The first capital repayment is just $PMT*(1+i)^{-n}$, and this grows geometrically with ratio $(1+i)$, i.e. at rate $i\%$ itself. Here the TVM sign convention is used, and END mode payments are assumed.

First set up n , $i\%$ and PMT (the original loan terms) in TVM, key in $v\%$ (the new valuation rate) and press $\boxed{R/S}$. The loan value at $v\%$ is displayed, $\boxed{\times\div}$ shows the value of the capital repayments(K), then $\boxed{-}$ gives the value of the interest repayments, where $K = -PMT * USPV(j\%,n)/(1+i)$ and $j\% = (v\% - i\%)/(1+i)$.

AMORT can do it accurately with a loop, separately discounting for the interest in each period. See "Lease vs. Purchase" page 49-53 of the *Solutions Handbook*, and the formula on page 143, showing the summation of the discounted cashflow and tax effects.

Keystrokes	Display	Keystrokes	Display	Keystrokes	Display
f P/R		%	07 - 25	RCL n	15 - 45 11
f CLEAR PRGM	00 -	+	08 - 40	y ^x	16 - 21
ENTER	01 - 36	÷	09 - 10	÷	17 - 10
ENTER	02 - 36	i	10 - 12	x [≥] y	18 - 34
RCL i	03 - 45 12	PV	11 - 13	i	19 - 12
-	04 - 30	g LSTx	12 - 43 36	PV	20 - 13
1	05 - 1	÷	13 - 10	g GTO 00	21 - 43,33 00
g LSTx	06 - 43 36	g LSTx	14 - 43 36	f P/R	

The "Rent or Buy Decision" pages 146-150 of the *Owner's Manual* uses AMORT at line 26 then divides the interest by n at lines 35 and 36. The formula on page 215 for PMT shows "interest" which is the average interest per period. This is a good first approximation, and really the only way if one wants to quickly solve for i - we need a regular repayment :-)

Example

10 n 10 i 1000 PV 0 FV PMT → -162.75. The total interest over the 10 years is given by: RCL n RCL PMT X RCL PV + → -430.66. This is what AMORT would give. Now, find the PV of this interest at 10%: 10 R/S → -339.19. We could calculate an income tax effect on this.

3. Pension Plan

In HP solver code: fsalmult=USFV((i%-f%)*SPPV(f%,1),t)*per%/100, where, at the end of each year, we pay in per% of the salary earned in that year over t years and earn i% on it. The salary inflates at f% in yearly steps. fsalmult gives the accumulated fund as a multiple of final salary (salary earned in the last year). Technically, the program should set END mode. BEG mode should give (1+i) times the END mode result, but here it gives (1+i)/(1+f) times - albeit the correct multiple of the new salary level, after the jump at time t, due to be earned in the next year. Here we just store t in n, i% in i, -per in PMT then key in f% and press R/S. Suppose we save 5% of salary over 25 years. Salary inflation is 3% and we earn 7% interest. What is our final salary multiple?

Keystrokes	Display	Keystrokes	Display	Keystrokes	Display
f P/R		g LSTx	05 - 43 36	PV	11 - 13
f CLEAR PRGM	00 -	%	06 - 25	FV	11 - 15
RCL i	01 - 43 12	+	07 - 40	g GTO 00	13 - 43, 33
x [≥] y	02 - 34	÷	08 - 10	f P/R	
-	03 - 30	i	09 - 12		
1	04 - 1	0	10 - 0		

25 [n] 7 [i], .05 [CHS] [PMT] 3 [R/S] → 2.05 times final salary. If we want 3 times final salary what do we need to save? 3 [FV] [PMT] → 0.073. How many years will it take to get to 4 times final salary? 4 [FV] [n] → 30. One can even re-solve for the "net" i% required to provide an outcome - this net figure is roughly just $i\% - f\%$ or, more precisely $(i\% - f\%) / (1 + f)$. So, this tiny program can be used to check on projections by investor advisors.

4. Graduated Payment Mortgages

This is 8 lines shorter than in the *Solutions Handbook* pages 32-34. It requires the 6 parameters to be stored in registers 1-6. General usage is as follows:

N [STO] 1 I% [STO] 2 L [STO] 3 B [STO] 4 G% [STO] 5 A [STO] 6 [R/S]

Where N, I%, and L are the loan term in years, interest rate (nominal annual), and amount. B and G% are the graduation term (years) and annual rate. A is the number of payments per year. The first result one gets is the very first instalment. See the example below. Other features are:

1. The speed does not depend on B, the number of years of increase. The book version is linearly time-dependent on B and surprisingly does not actually *use* the more general formula on the last page (152) - instead it uses an elegant loop to calculate PMT1. One could say it uses this in terms of solver code:

$$L = \text{PMT1} * (\text{SIGMA}(H:0:B-1:1: \text{USPV}(I\%/A, A) * \text{SPPV}(I\%/A, H * A) * \text{SPFV}(G\%, H)) + \text{USPV}(I\%/A, (N-B) * A) * \text{SPPV}(I\%/A, B * A) * \text{SPFV}(G\%, B))$$

Whereas, here we implement the formula directly:

$$L = \text{PMT1} * \text{FV}(B, Q\%, \text{USPV}(I\%/A, (N-B) * A), \text{USPV}(I\%/A, A), 1, 0)$$

where $Q\% = \text{SPPV}(I\%/A, A) * (G\% - I\%/A * \text{USFV}(I\%/A, A))$. First, Q% is calculated at line 17 (and stored in [i] at line 34). The second USPV above is calculated at line 23 (and stored in [PMT] at line 32), the first USPV at line 30 and the [FV] at line 37. Thereafter (lines 47-69) [AMORT] is used to amortise the loan. Because [AMORT] stores the *rounded* [PMT] we must keep the full precision payment separate (in R7).

2. A case can be re-run simply by pressing [f] CLEAR [PRGM] [R/S]. The book version requires re-entry of all data.

3. This version allows a variable number of payments per period. The book program is for *monthly payments only* and yearly increase steps. With this program we could have bi-annual increases for example, with quarterly payments.

4. The book version only shows balances up to point B, so one never gets to see how the balance starts to decrease. Here we can continue amortising past this point.

Example from page 35 of the *Solutions Handbook*

[f] 2 30 [STO] 1 12.5 [STO] 2 50000 [STO] 3 5 [STO] 4 5 [STO] 5 12 [STO] 6
 [R/S] → 1.00 [R/S] → -448.88 [R/S] → -50,914.67 [R/S] → 2.00 [R/S] → -471.33 [R/S] → -51,665.08
 [R/S] → 3.00 [R/S] → -494.89 [R/S] → -52,512.33 [R/S] → 4.00 [R/S] → -519.64 [R/S] → -52,523.86
 [R/S] → 5.00 [R/S] → -545.62 [R/S] → -52,547.98 [R/S] → 6.00 [R/S] → -572.90 [R/S] → -52,217.84
 [R/S] → 7.00 [R/S] → -572.90 [R/S] → -51,849.65 [R/S] → 8.00 [R/S] → -572.90 [R/S] → -51,432.70

From year 6 on the PMT is constant and the loan balance reduces. The book has three balances 0.01 different - here we use $\boxed{\text{AMORT}}$, whereas the book uses $\boxed{\text{FV}}$.

Keystrokes	Display	Keystrokes	Display	Keystrokes	Display
$\boxed{\text{f}} \boxed{\text{P/R}}$		$\boxed{\text{PV}}$	23 - 13	$\boxed{\text{PMT}}$	47 - 14
$\boxed{\text{f}} \boxed{\text{CLEAR}} \boxed{\text{PRGM}}$	00 -	$\boxed{\text{RCL}} \boxed{1}$	24 - 45 1	$\boxed{\text{STO}} \boxed{7}$	48 - 44 7
$\boxed{\text{f}} \boxed{\text{CLEAR}} \boxed{\text{FIN}}$	01 - 42 34	$\boxed{\text{RCL}} \boxed{4}$	25 - 45 4	$\boxed{1}$	49 - 1
$\boxed{\text{g}} \boxed{\text{END}}$	02 - 43 8	$\boxed{-}$	26 - 30	$\boxed{\text{STO}} \boxed{+} \boxed{0}$	50 - 44 40 0
$\boxed{\text{RCL}} \boxed{2}$	03 - 45 2	$\boxed{\text{RCL}} \boxed{6}$	27 - 45 6	$\boxed{\text{RCL}} \boxed{0}$	51 - 45 0
$\boxed{\text{RCL}} \boxed{6}$	04 - 45 6	$\boxed{\text{X}}$	28 - 20	$\boxed{\text{R/S}}$	52 - 31
$\boxed{\text{n}}$	05 - 11	$\boxed{\text{n}}$	29 - 11	$\boxed{\text{RCL}} \boxed{\text{PMT}}$	53 - 45 14
$\boxed{\div}$	06 - 10	$\boxed{\text{PV}}$	30 - 13	$\boxed{\text{R/S}}$	54 - 31
$\boxed{\text{i}}$	07 - 12	$\boxed{\text{R}} \boxed{\downarrow}$	31 - 33	$\boxed{\text{RCL}} \boxed{6}$	55 - 45 6
$\boxed{\text{STO}} \boxed{0}$	08 - 44 0	$\boxed{\text{PMT}}$	32 - 14	$\boxed{\text{f}} \boxed{\text{AMORT}}$	56 - 42 11
$\boxed{\text{CHS}}$	09 - 16	$\boxed{\text{R}} \boxed{\downarrow}$	33 - 33	$\boxed{\text{RCL}} \boxed{\text{PV}}$	57 - 45 13
$\boxed{\text{PMT}}$	10 - 14	$\boxed{\text{i}}$	34 - 12	$\boxed{\text{CHS}}$	58 - 16
$\boxed{\text{FV}}$	11 - 15	$\boxed{\text{RCL}} \boxed{4}$	35 - 45 4	$\boxed{\text{R/S}}$	59 - 31
$\boxed{\text{RCL}} \boxed{5}$	12 - 45 5	$\boxed{\text{n}}$	36 - 11	$\boxed{\text{RCL}} \boxed{4}$	60 - 45 4
$\boxed{-}$	13 - 30	$\boxed{\text{FV}}$	37 - 15	$\boxed{\text{RCL}} \boxed{0}$	61 - 45 0
$\boxed{\text{FV}}$	14 - 15	$\boxed{\text{RCL}} \boxed{0}$	38 - 45 0	$\boxed{\text{g}} \boxed{\text{x} \leq \text{y}}$	62 - 43 34
$\boxed{\text{CLx}}$	15 - 35	$\boxed{\text{i}}$	39 - 12	$\boxed{\text{g}} \boxed{\text{GTO}} \boxed{65}$	63 - 43,33 65
$\boxed{\text{PMT}}$	16 - 14	$\boxed{\text{CLx}}$	40 - 35	$\boxed{\text{g}} \boxed{\text{GTO}} \boxed{49}$	64 - 43,33 49
$\boxed{\text{PV}}$	17 - 13	$\boxed{\text{STO}} \boxed{0}$	41 - 44 0	$\boxed{\text{RCL}} \boxed{7}$	65 - 45 7
$\boxed{1}$	18 - 1	$\boxed{\text{n}}$	42 - 11	$\boxed{\text{RCL}} \boxed{5}$	66 - 45 5
$\boxed{\text{CHS}}$	19 - 16	$\boxed{\text{RCL}} \boxed{3}$	43 - 45 3	$\boxed{\%}$	67 - 25
$\boxed{\text{PMT}}$	20 - 14	$\boxed{\text{PV}}$	44 - 13	$\boxed{+}$	68 - 40
$\boxed{\text{CLx}}$	21 - 35	$\boxed{\text{x} \geq \text{y}}$	45 - 34	$\boxed{\text{g}} \boxed{\text{GTO}} \boxed{47}$	69 - 43,33 47
$\boxed{\text{FV}}$	22 - 15	$\boxed{\div}$	46 - 10	$\boxed{\text{f}} \boxed{\text{P/R}}$	

TVM runs like the wind on the *new* 12c platinum (*12cpt*). You don't even see "running" when it solves for $\boxed{\text{n}}$, $\boxed{\text{PV}}$, $\boxed{\text{PMT}}$ or $\boxed{\text{FV}}$ and $\boxed{\text{AMORT}}$ is easily ten times faster over 200 payments for example. Also the *12cpt* doesn't store the rounded $\boxed{\text{PMT}}$ (it uses the rounded $\boxed{\text{PMT}}$ though) while doing $\boxed{\text{AMORT}}$, so the extra R_7 is not required. A *12cpt* RPN version requires 4 extra lines. The $\boxed{\text{RCL}} \boxed{7}$ at line 65 becomes $\boxed{\text{RCL}} \boxed{\text{PMT}}$. The $\boxed{\text{STO}} \boxed{7}$ at line 48 is not required. $\boxed{\text{R}} \boxed{\downarrow}$ is needed before the $\boxed{\text{RCL}} \boxed{3}$ at line 43. And both $\boxed{\text{PV}}$ s at lines 23 and 30 need to be $\boxed{\text{R}} \boxed{\downarrow} \boxed{\text{PV}} \boxed{\text{PV}}$ or $\boxed{\text{PV}} \boxed{\text{x} \geq \text{y}} \boxed{\text{R}} \boxed{\downarrow}$ - on the *12cpt* TVM input does not disable stack lift. If you run the above example in FIX 9 instead of FIX 2 and after year 6 do $\boxed{\text{RCL}} \boxed{1} \boxed{\text{RCL}} \boxed{6} \boxed{\text{X}} \boxed{\text{RCL}} \boxed{\text{n}} \boxed{-} \boxed{\text{f}} \boxed{\text{AMORT}}$ $\boxed{\text{RCL}} \boxed{\text{PV}} \boxed{\text{CHS}}$ on both the 12C and *12cpt* you will notice how much faster the *12cpt* is at $\boxed{\text{AMORT}}$, and also see -0.000005479 on the *12cpt* and 0.002015600 on the 12C. The *12cpt* is a good 400 times closer to zero - it uses a slightly lower PMT and should be called the 12d as it has 12 significant digits under the hood!! Indeed, $3 \boxed{\frac{1}{x}} \boxed{1000} \boxed{\text{X}} \boxed{\text{g}} \boxed{\text{FRAC}}$ gives 0.333333300 on the 12C and 0.333333333 on the 12d. Maybe the next version will have a 12 digit display too :-)